Diversification Optimization

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Revision date: October 5, 2007

Diversification Optimization was previously referred to as
CVDRAM(Correlated Vector Dominated Region Allocation Model)

Abstract:

Diversification Optimization is a holistic asset allocation model and visualization platform. It maps asset correlations to vector angles and projects them into three dimensions. Vector lengths define the relative attractiveness of assets by measures such as the Sharpe Ratio. The model imposes a convex hull on the asset XYZ pointcloud. Assets comprising the convex hull create the efficient set. Assets dominated by the convex hull are inefficient assets. The total portfolio value equals the volume of the polyhedron. The model gives an optimal asset allocation by dividing each assets' pro rata volume by the polyhedron volume. Assets, vectors, and the convex hull are displayed on a three dimensional graph.

Diversification Optimization uses the statistical components employed by traditional mean-variance optimization (MVO) See Markowitz [1952]; however, it utilizes these portfolio metrics in a dramatically new fashion that yields unique output. The model gives mangers new insight and analysis capabilities and is thus an entirely new model as opposed to a refinement of the MVO paradigm.

Diversification Optimization portfolios maximize diversification and then adjust the asset weightings to account for differences in the attractiveness of each asset. A utility function such as the Sharpe ratio measures each asset's relative attractiveness. See Sharpe [1970].

Diversification Optimization models diversification as a trans-dimensional factor. It represents diversification in every dimension, not in any single dimension.

The combination of all assets graphed simultaneously creates the optimal depiction of portfolio balance. The model geometrically equates portfolio diversification to physical balance, represented as symmetry in three dimensions.

Diversification Optimization

This pictorial tour shows a two-dimensional example of Diversification Optimization. Three-dimensional Diversification Optimization creates a projection of a portfolio optimized for visual acuity. The process is much like that of a cartographer that creates a two dimensional map of our three dimensional world.

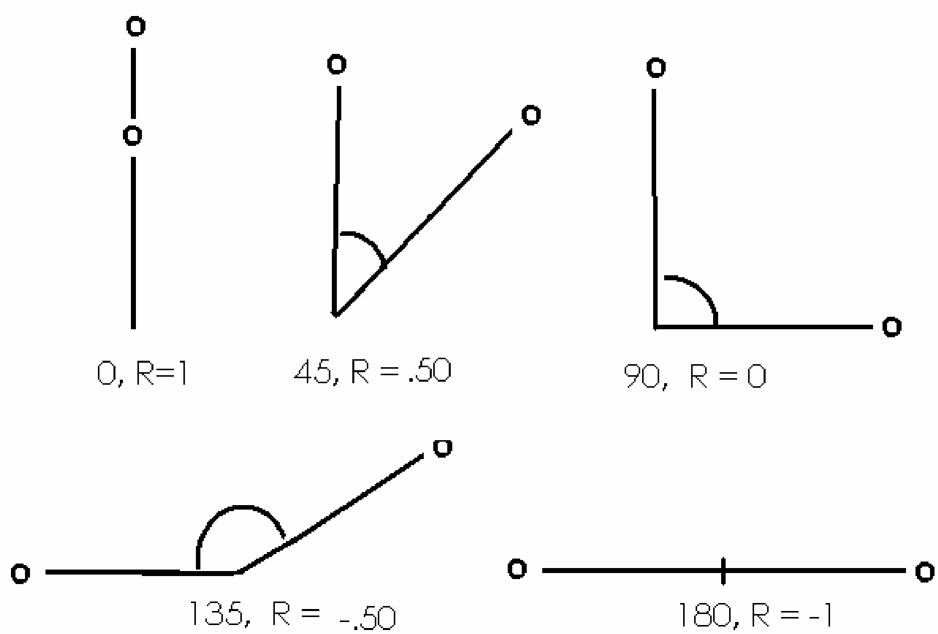
The model reduces the portfolios dimensionality using a genetic optimization. The genetic optimization is necessary as traditional optimization methods fail given the non-linear composition. See Goldberg [1989]. A portfolio has an original dimensionality equal to the number of assets that it contains, but requires an approximation to reduce the dimensionality for visualization purposes. Diversification Optimization reduces that dimensionality to obtain visual insight and more easily understood output.

This process gives every asset a vector that geometrically explains the relative uniqueness of each asset in the portfolio. Diversification Optimization assigns assets that share strong relationships (high correlations) with each other vectors with small angles separating their position. It considers no angle in excess of 180 degrees, as we express the relationship as the minimum angle in any available plane. For computational simplicity, we take the cosines of the angles.

The model equates correlations to cosines of angles for every asset in the space. This relationship is show in Exhibit 1.

1 correlation = 0 degree separating angles
0 correlation = 90-degrees separating angles
-1 correlation = 180 degrees separating vectors
R = Correlation

Exhibit 1
Relationship between angle and correlation



Additionally, due to the implementation of randomness within the genetic algorithm, the results can vary even with the same inputs. This randomness is necessary to push the evolutionary search past local minimums. In three dimensions, it is unlikely to produce a portfolio with no error or distortion. Diversification Optimization therefore provides excellent and robust solutions, but does not assure a perfect solution. This distortion of the result is rather forgivable in that correlations exhibit a self-correcting mechanism. Such that given Correlation AB and correlation AC we can better imply correlation BC. Further, an asset allocation decision is best a forecasting endeavor. Indeed, it is better to be wrong but close than to be perfect but impractical.

Because the genetic algorithm is minimizing the difference between known correlations and graphed angles of asset coordinates, some distortion is likely. For each unique element in a known correlation matrix, the model adds an error allowance variable. Errors may be either positive or negative since they depict the difference between the actual correlations and the measured angles separating the vectors. Squaring the errors provides positive values and helps ensure a more even distribution of error accumulation.

The amount of distortion is a function of the original dimensionality of the correlation matrix. For example, a 25-asset correlation matrix will have 25 dimensions. To guarantee a perfect mapping we need 25 dimensions to calculate the results. Unfortunately, all geometric and visual intuition is lost at dimensions greater than three. In cases of large portfolios, we can compute Diversification Optimization to obtain the optimal allocation results, but either forego or decouple the visualization as the vector mapping and volume calculations take place in high dimensions.

The genetic algorithm generates trial solutions by iterating X, Y, and Z data. Each XYZ data point represents a location of an asset.

The algorithm creates three matrices of equal size. Equation 1 depicts their relationship to one another.

C: an estimated correlation matrix, generally derived from historical sample data
A: a matrix of the cosines between vectors graphed in N space
E: a error matrix that gives the difference between C and A

C + E = A
(1)

The genetic algorithm begins with a population of trial XYZ points for all asset candidates. Equation 2 provides the fitness function for evaluating the correctness of this population of points.

$$\sum_{j=1}^{n} \sum_{i=1}^{n} Eij^2$$

(2)

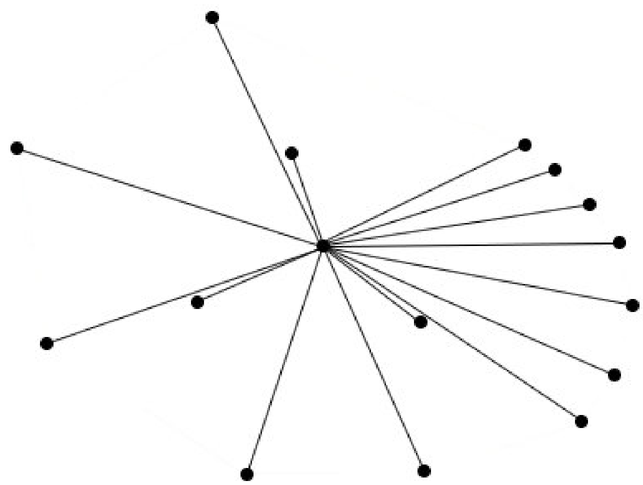
E is the error matrix above.

Through crossover, mutation, and regeneration, successive generations of trial solutions minimize the fitness function. See Whitley [2001]. After sufficient iterations the fitness function convergences, and the model uses the corresponding XYZ coordinates.

The combination of all assets graphed simultaneously creates the optimal depiction of portfolio balance. Diversification Optimization maps portfolio diversification to physical balance, represented as symmetry in three dimensions. Rotational symmetry, central symmetry and mirror symmetry all apply. See Weisstein "Symmetry."

Exhibit 2 is a depiction of a number of asset vectors. You can see how the assets graphed to the left are relatively more unique than the assets graphed on the right. Those on the right will crowd one another out fighting for the same allocation space. Consequently, the points to the left will capture a larger relative portion of the volume.

Exhibit 2
Depiction of assets described in Diversification Optimization framework



The origin of the graph is a hurdle rate any candidate must beat for consideration in the portfolio. It is usually the greater of the return on a risk-free asset or money market fund, the interest rate paid for leveraged portfolios or a performance benchmark. The origin can also be set to the inflation rate, implying that the investor requires a positive real rate of return. The risk-free rate dominates risky assets with returns inferior to the risk-less asset. Diversification Optimization summarily excludes such assets.

The model considers only investment candidates with a positive return expectation. Therefore, the model must aggregate any position hedge with a negative return expectation with the other component(s) of the hedge prior to inclusion. This is consistent with the business logic initiating the original hedged position.

As shown in Exhibit 2, vectors have various lengths. The length of any vector is a utility function that quantifies the relative attractiveness of any asset. The greater the utility function's value the longer the vector length.

Diversification Optimization normalizes the asset coordinates provided by the optimization to unit vectors and multiplies the vectors by the utility function.

A vector length of equivalent distance from the origin (in any direction) will have the same utility function value. Diversification Optimization uses correlations or diversification as a trans-dimensional factor. Therefore, each X, Y and Z-axis is measuring the same utility function, and the relative position for each asset is an explanation of diversification. Due to the absolute relativity of the model, there is no difference between one model polyhedron and that models' mirror opposite. In other words, the set position of the model is arbitrary. Only the models intra-relationships are meaningful.

The Sharpe ratio is a basic measure of utility. The Sharpe ratio is a value that takes the expected return of an asset (in excess of a risk-free return) and divides that value by the expected risk of that asset (normally measured by the assets' standard deviation). This function gives the model a reasonable method for comparing assets. Creator Professor William Sharpe won a share of the 1991 Nobel Prize for Economics for his related work. See Sharpe, [1970]. Diversification Optimization selects the Sharpe ratio, Sortino ratio or Calmar ratio as the utility function. Fixed income investors can substitute some quantitative model of credit risk as the risk measurement. The optimal risk definition is one that most accurately independently defines the risk of the asset.

Diversification Optimization is fungible with a variety of statistics used as a utility function. It can substitute other values such as credit ratings, rankings, or proprietary algorithms in lieu of the pure risk-adjusted returns. For example, it can use only returns, only risk, or return / risk. In the equity and fund arena, conventional risk computations are standard deviation, maximum drawdown, and semi-variance. See Markowitz [1959]. These three risk definitions determine the Sharpe ratio, The Calmar Ratio and the Sorintino Ratio respectively. Any of the three metrics are reasonable utility functions.

The flexibility inherent in using the utility function as vector length enables the modeler to construct the portfolio most indicative of the real risks and opportunities of the investment candidates going forward. This flexibility can manifest not only in the election of risk, but also in the relative significance of risk, return and diversification. For example, it may be appropriate to normalize the individual asset risk levels. This would create a portfolio with asset allocation results that are only dependant upon returns and diversification. To make an asset allocation solely dependant upon diversification, shrink both risk and return estimations to the population mean. Using different methods to condition or scale the utility function are therefore a means to determine the relative significance of that variable in the model. In a pragmatic sense, confidence in the estimated values is the primary criteria for determining the need and magnitude of data conditioning.

One simple and consistent tool to condition historical data for future pertinence is using a James-Stein Estimation. See James [1960].

Historical returns are generally inferior to historical risk or correlations in their predictive value. See Sharpe [1990]. Therefore, shrinking the range of returns and diminishing the anomalies of the past can be an excellent and simple tool to make the asset allocation models pertinent for forward looking endeavors. Scaling the range of data is often appropriate, especially when the historical data is from individual assets or from short sample periods. A complete normalization of both risk and return data to the population mean results in a portfolio allocating to every asset expect possibly cash. This portfolio is similar to the maximum diversification portfolio and only differs to the extent of the deviation of the approximation.

Greater distances signify that the asset is more likely to extend the shell of the model, dominate its peers, and capture a greater proportion of volume.

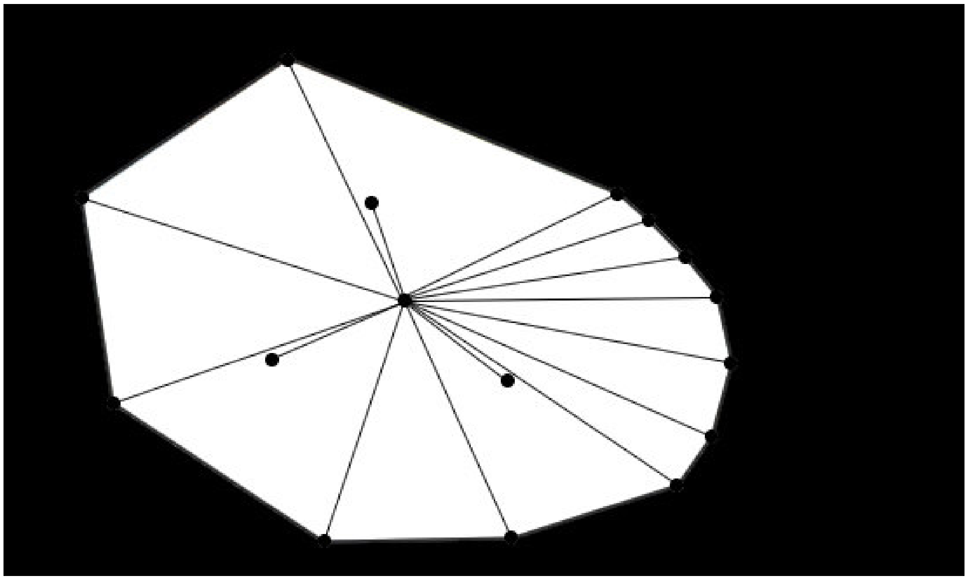
Applying a James-Stein Estimation can also instruct the model to weight the relative importance on assets' risk or return. For example, in a well-diversified portfolio, more of the portfolio risk can be mitigated at the portfolio level visa vie diversification. In such cases, especially in concert with an absolute return objective, using a higher level of risk shrinkage relative to the return shrinkage will have the effect of decreasing the variances of returns and diminishing the importance of the individual assets' risk levels when determining the optimal asset allocation model.

Risk can be mitigated at the position level using standard position management techniques like stop loss orders. Risk management also manifests at the candidate selection step. Therefore, it can be very practical to diminish the relative importance of position-specific risk, as there are other opportunities for the management of total risk.

Asset allocation is about using historical data to derive forward-looking expectations. A flexible system of scaled utility functions rooted in diversification assures fiduciary portfolio construction.

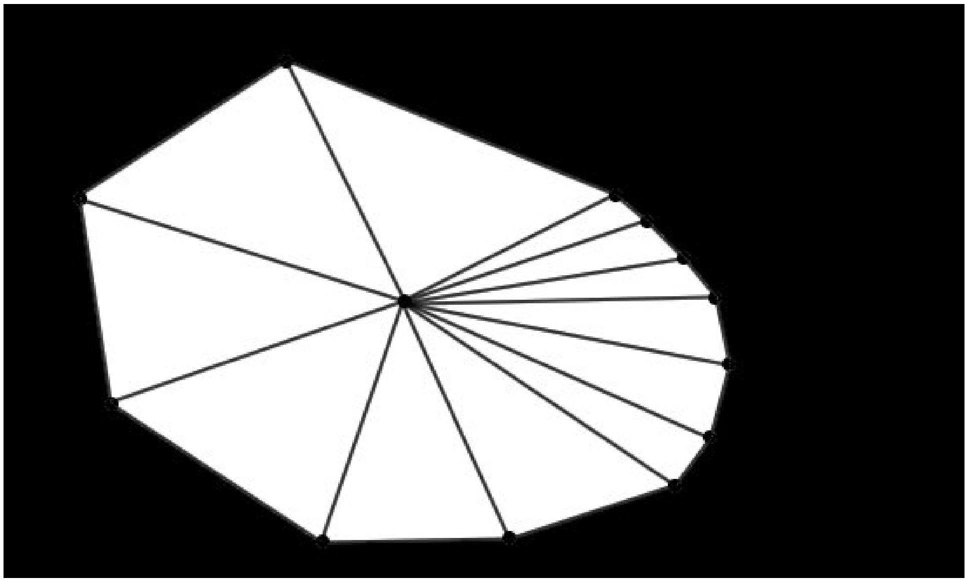
We connect the assets comprising the convex hull in Exhibit 3, clearly showing three dominated assets.

Exhibit 3
Efficient and non-efficient assets



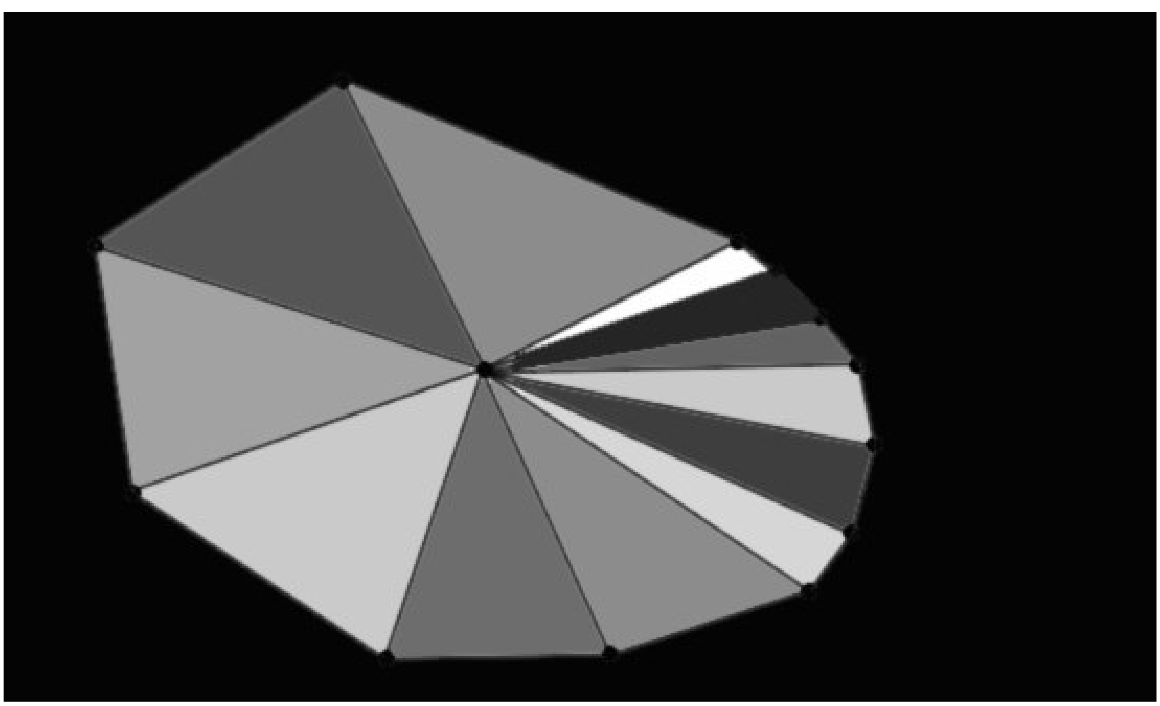
The enclosure imposed on the pointcloud is a convex hull. See O'Rourke [1998]. Whereas MVO creates an efficient frontier, the convex hull is analogous to the frontier, but comprised of only the efficient assets in the portfolio. Therefore, the convex hull is an efficient frontier of individual assets comprising one single efficient portfolio. The shape of the hull depicts the balance of the portfolio. This hull also determines the volume of the model portfolio, which is set equal to the capital allocated. Diversification Optimization only uses the interior space for its calculations. Essentially, everything inside is the portfolio.

Exhibit 4
Diversification Optimization efficient asset portfolio



Diversification Optimization removes the interior assets as shown in Exhibit 4. Assets comprising the convex hull dominate assets interior to the model. Allocations to such assets will drag down portfolio performance and are therefore inefficient. Diversification Optimization will not allocate to these interior or dominated assets. Diversification Optimization literally casts dominated assets in the shadow of superior assets. This is because there is always a combination of two or more assets that provide superior performance.

Exhibit 5
Diversification Optimization allocation scheme



As shown in Exhibit 5 the optimal allocation depends on both the distance from the origin, and the relative uniqueness of that asset. This uniqueness explains how the asset increases portfolio diversification. Notice the assets on the right crowd out one another for the same space.

In this example, as well as other well-balanced portfolios, we delete the dominated risk-free asset. However, it is common for the cash (risk-free) asset to be included in the composition of the convex hull. When this happens the cash asset actually increases the volume of the model and becomes an efficient asset. The model assumes risk-free assets to have zero correlation with any other asset. Subsequently, allocations to cash can increase portfolio diversification, despite often having poor expected returns.

According to Diversification Optimization, any portfolio consisting of one asset and a risk-free asset will allocate equally to the two, as both assets contribute half of the two-dimensional line segment.

Diversification Optimization allocates resources based on pro-rata volume. To determine each region's volume we first calculate the center of gravity or "centroid" for the model. See Weisstein "Geometric Centroid."

The center of mass is $O = (x_1, x_2, x_3)$. To find its coordinates, x_i, we use Equation 3

$$x_i = \frac{\sum_{j=1}^{N} x_{i,j}}{3}$$

(3)

Where *N* is the number of assets and $x_{i,j}$ is an i-th position, coordinate of the j-th asset.

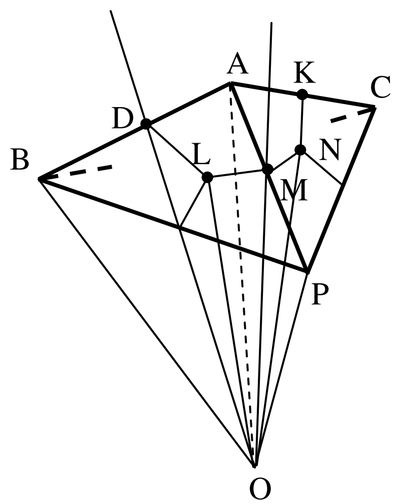
Next, Diversification Optimization splits the asset comprising the convex hull with bisecting plane. The model creates a facet with asset as the vertex and all of the adjacent bisecting planes. This facet connects to the model centroid to create the assets' unique relative volume. Exhibit 6 illustrates the splitting where

A, B, C and P are assets;
O is the center of mass;
ABP and APC are facets of the convex hull

L and N are the centers of the facets.

The model calculates the using Equation 3 with *N=3* and *j* is a number of the vertex of the facet OD and OM are the bisecting planes of the BOA and POA angles.
Thus, the bounding bisecting planes are ODL, OLM, OMN, OMK etc.

Exhibit 6
Asset vertices, facets and bisecting vectors.

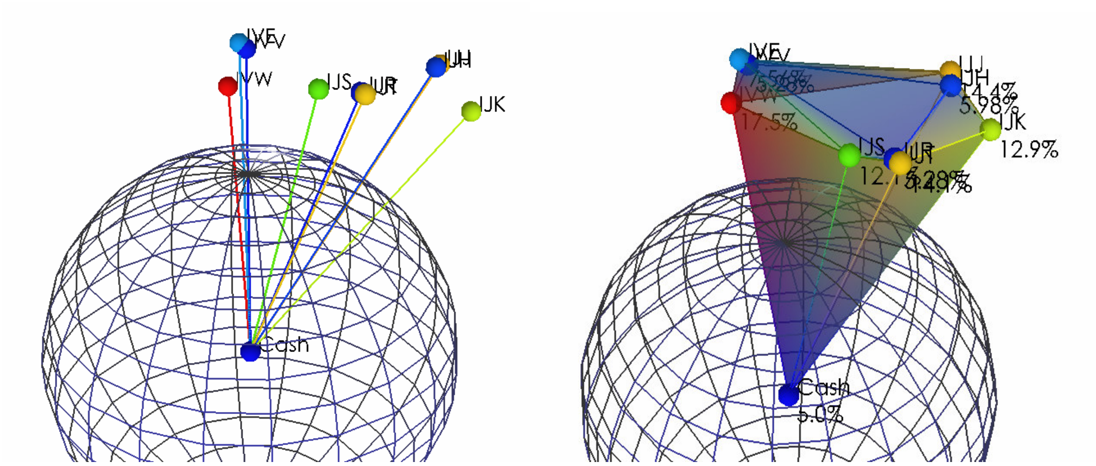


Finally, the pro-rata volume is calculated. The area of the convex hull is calculated, and then Diversification Optimization calculates the volume for each region (asset) and determines the ratio of any region relative to the entire volume. This provides an optimal allocation for each asset.

Exhibit 7 shows three portfolios captured from the software. The overlaid globe represents perfect symmetry and allows for easy comparison of one portfolio to another.

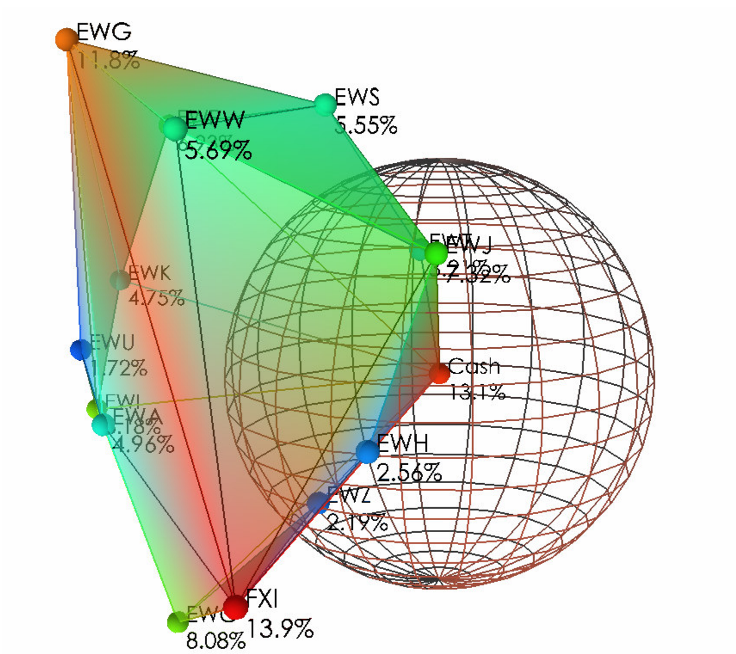
Exhibit 7
Example portfolios

Portfolio 1 shows an unbalanced asset allocation built form 9 Barclays 'Style Box' Exchange Traded Funds.



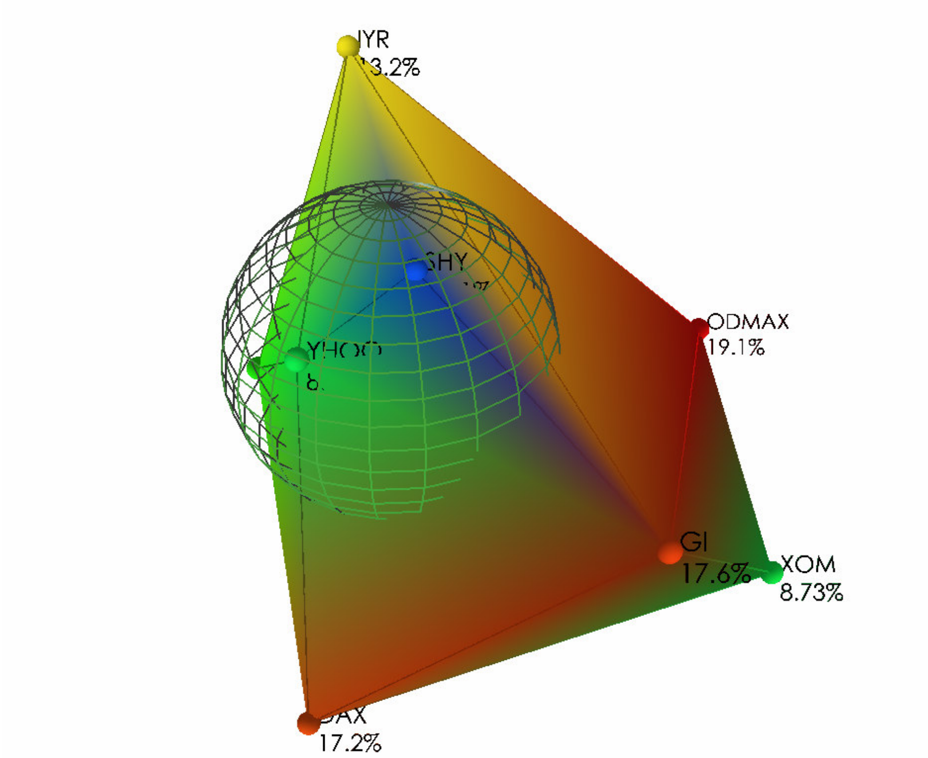
Portfolio 1

Portfolio 2 is built from 19 Barclay's I-Shares Each representing a different country. The diversification is approaching 50% but not quite there. A vertical bisector would show that this portfolio exists entirely on the left side, therefore demonstrating less than 50% of potential diversification.



Portfolio 2

Portfolio 3 is an 18-asset portfolio built from a variety of instruments such as stocks, futures, mutual funds and exchange traded funds, each representing an asset class. Here the symmetry exhibited comes closer to covering more of the area depicted by the overlaid globe.



Portfolio 3

Simulation

Mimicking our universe, we can apply time as a fourth dimension. Using the fourth dimension, the model can display results of simulations or changes to a portfolio by statistical ranking, chronology, or random display based on a Monte Carlo type distribution. See Rubinstein [1981].

Since the dimensionality is the only source of error in the model, increasing the dimensionality without sacrificing visual intuition is a primary goal. In this case, using time as the fourth dimension gives the user greater flexibility. Diversification Optimization can apply a Monte Carlo type simulation to the model's inputs. The inputted annual return is set to the mean return and the model simulates the correlations with both values deviating from the original or mean value by a probability distribution given by the assets individual standard deviation. For the simulation of correlations, the original correlation is set to the mean and the model creates simulated portfolios by sampling random points in a covariance distribution given by the mean correlation and asset co variances.

This process creates "N" number of optimized portfolios with the simulated inputs. Taking a simple average of the allocation results yields a portfolio superposition. This superposition results in an optimal portfolio in four dimensions. The simulation is especially useful for smoothing out allocation results within the candidate set. The most sensitive assets are those inefficient assets dominated by assets with large volume allocations. The simulation reveals that such assets can in fact be efficient given the perturbations inherent in the performance of investment candidates.

Other deeply dominated assets seldom appear efficient. The portfolio superposition minimizes such assets' allocations.

More distortion occurs as the dimensionality of the original matrix increases. Portfolios with greater than approximately 50 assets deteriorate until the result is no longer statistically significant. In these cases, diversification presented in three or four dimensions appear visually overstated and the allocation results are sub-optimal.

Diversification Optimization and Portfolio Theory

Modern Portfolio Theory (MPT) judges a portfolio of investments by two characteristics: risk and return. The Efficient Frontier created by Markowitz graphs the risk and return composition of portfolios. See Markowitz [1952]. Risk and return are certainly the correct parameters for accessing any one security or any one portfolio. For portfolio analysis, this is the definitive relative methodology. MPT uses diversification behind the scenes to reduce risk. The relative methodology has inherent drawbacks best complemented with a holistic representation. Any holistic representation requires information about the relationships of the portfolio components. Diversification Optimization captures this requirement by correlating and representing each asset relationship.

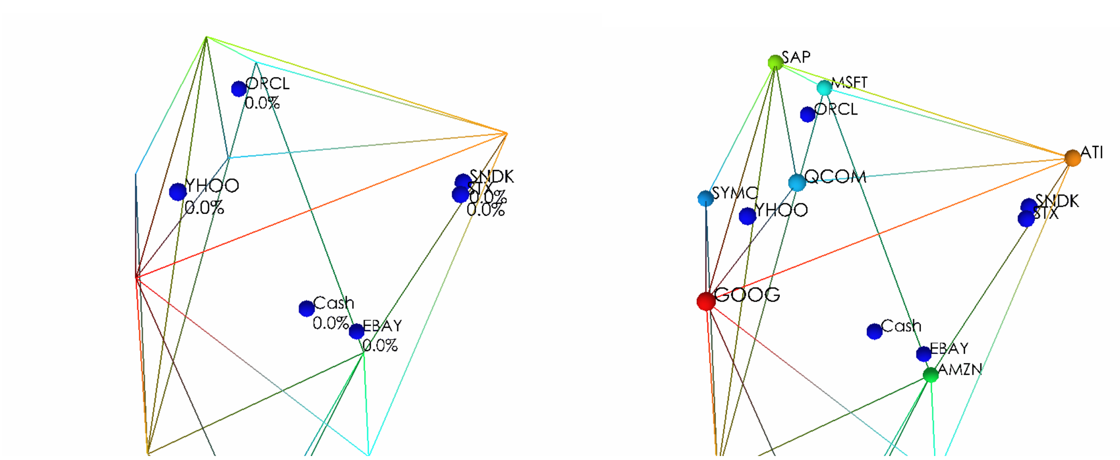
Diversification Optimization creates one single globally optimal portfolio, rather than the spectrum of risk-efficient portfolios created by mean-variance optimization. A mean-variance efficient portfolio is graphically depicted as a single point juxtapose other mean-variance efficient portfolios and individual assets, Diversification Optimization produces a holistic depiction of a portfolio that graphs all portfolio components and taken together create the model portfolio. This provides a new perspective on a portfolio that gives insight to the portfolio's internal dynamics.

Application:

Both MVO and Diversification Optimization create efficient portfolios. Owning an efficient portfolio should be a basic expectation for any investor. Yet few investors attain this critical distinction. An efficient portfolio is one that best fulfills the objectives of the investor while minimizing the resources required producing the results. This broad description can take many embodiments. For some, an efficient portfolio maximizes return for a given level of risk. For this, the MVO framework is sufficient. However, for others who require a portfolio to maximize returns, net present value or terminal wealth, Diversification Optimization is a more practical tool for such portfolio objectives as it enforces diversification and reduces the concentration and model risk associated with MVO portfolios engendered to maximize returns. A convex hull is a geometrically efficient model. Convex hulls maximize the volume of the model while conserving the energy of the boundary surface. This is true for all piecewise smooth bodies of the same volume.

Diversification Optimization's visualization capacity provides visual intuition of the sensitivities of any allocation. Hence, one can visualize the domination sensitivities of any efficient or inefficient asset. Additionally, the user can determine the sensitivities of the asset allocation model to either changes in the return to risk ratio or changes in the correlations assumptions.

Exhibit 8
An Illustration of the inefficient assets (left) and the assets that dominate them (right)



Efficient portfolios are not created equal. They depend on the objectives, intentions, preferences and expectations of the investor. These qualities determine the investment policy and strategy, the selection of investment candidates, and the performance expectations for those candidates.

In Diversification Optimization, these considerations manifest in the universe selection process. Selecting a sufficient range of assets will help ensure that the portfolio will meet future liquidity requirements. It is appropriate to verify that the optimal allocation will indeed meet liquidity needs. However, once these requirements are satisfied, Diversification Optimization ensures the manager has created a suitable, prudent portfolio since Diversification Optimization produces one globally efficient portfolio.

A globally efficient portfolio eliminates the needs for risk profiling to match customers with portfolios optimized for specific levels of standard deviation. This avoids the perils of psychometric mapping of investors to certain risk levels when the advisor or investor poorly understands risk.

Summary

The Diversification Optimization portfolio condenses all of the information in a correlation matrix into an intuitive three-dimensional model portfolio. It considers correlations of assets along with risk and return or custom technical, statistical and fundamental information to provide the investor with a complete picture of their portfolio's opportunities, strengths and weaknesses.

Explicit correlation visualization gives the investment manager more control and inherently demonstrates the benefits of diversification-enhancing investments such as emerging markets and alternative investments.

Diversification Optimization provides the first holistic portfolio visualization and asset allocation model. Highlighting correlation and promoting diversification enables investors to clearly assess and communicate portfolio risk and diversification.

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